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Radion on the de Sitter brane

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The radion on the de Sitter brane is investigated at the linear perturbation level, using the covariant curvature tensor formalism developed by Shiromizu, Maeda and Sasaki [1]. It is found that if there is only one de Sitter brane with positive tension, there is no radion and thus the ordinary Einstein gravity is recovered on the brane other than corrections due to the massive Kaluza-Klein modes. As a by-product of using the covariant curvature tensor formalism, it is immediately seen that the cosmological scalar, vector and tensor type perturbations all have the same Kaluza-Klein spectrum. On the other hand, if there are two branes with positive and negative tensions, the gravity on each brane takes corrections from the radion mode in addition to the Kaluza-Klein modes and the radion is found to have a negative mass-squared proportional to the curvature of the de Sitter brane, in contrast to the flat brane case in which the radion mass vanishes and degenerates with the 4-dimensional graviton modes. To relate our result with the metric perturbation approach, we derive the second order action for the brane displacement. We find that the radion identified in our approach indeed corresponds to the relative displacement of the branes in the Randall-Sundrum gauge and describes the scalar curvature perturbations of the branes in the gaussian normal coordinates around the branes. Implications to the inflationary brane universe are briefly discussed.

I. INTRODUCTION

Based on the idea of a brane-world suggested from string theory [2], Randall and Sundrum proposed an interesting scenario that we may live on a 3-brane embedded in the 5-dimensional anti-de Sitter space [3,4]. One of the attractive features of their scenario is that gravity on the brane may be confined within a short distance from the brane to be effectively 4-dimensional even for an infinitely large extra-dimension [4,5]. This occurs on a brane with positive tension, and it is because the AdS bulk on both sides of the brane shrinks exponentially as one goes away from the brane.

In their comprehensive paper [5], Garriga and Tanaka has shown that if there is only one flat brane with positive tension, the zero-mode truncation, which means to discard all the excitation modes except for the one with the lowest (zero) eigenvalue, leads to 4-dimensional (linearized) Einstein gravity on the brane. On the other hand, if there are two flat branes of positive and negative tensions, the zero-mode truncation leads to a Brans-Dicke type theory with positive and negative Brans-Dicke parameters, respectively, on the branes with positive and negative tensions. The difference can be understood by considering the role of the so-called radion mode, or the unique non-trivial scalar mode of gravitation in the vacuum bulk spacetime. (Hereafter, we use the term, scalar, in the sense of the 4-dimensions unless otherwise noted.) The radion mode corresponds to displacement of the brane location [5,6]. If only a single flat brane is present, it is impossible to know locally if the brane is displaced because of the general covariance. Hence, the radion plays no role in the gravity on the brane. However, if two flat branes are present, the relative motion of the branes, or the relative displacement of the branes is physical. Therefore the radion mode is expected to affect the gravity on the brane. Thus if we live on a single positive-tension brane, the effect of the extra-dimension appears only through the non-zero-mode excitations, or the so-called Kaluza-Klein modes.

However, the above interpretation is given to the case of flat Minkowski branes. It is therefore interesting to ask if the same argument applies to the case of cosmological brane models, particularly when the universe is spatially closed. It was suggested that a spatially closed, inflating brane-universe may be created from nothing [7]. In this model, the universe is described by the de Sitter brane embedded in the 5-dimensional anti-de Sitter space. Since this brane-universe is spatially closed, the curvature radius of the universe becomes indeed the 5-dimensional radius of the

brane. Then, one may expect that the radion, which describes fluctuations of the brane location, becomes physical even in the case of a single brane, and contributes to gravity on the brane.

In this paper, we carefully investigate perturbations of the de Sitter brane-world and clarify this issue of the cosmological radion. Cosmological brane perturbations have been studied by various authors with metric-based approaches [7–19]. We adopt the covariant curvature tensor formalism developed by Shiromizu, Maeda and Sasaki [1]. A covariant approach to cosmological brane perturbations based on this formalism has been developed by Maartens [20]. In this formalism, the effective gravitational equation is written in the form of the 4-dimensional Einstein equations with a couple of modifications on the right-hand-side. Namely, there appear a tensor quadratic in the matter energy-momentum tensor and an "electric" projection of the 5-dimensional Weyl tensor. Since the former is locally described by the 4-dimensional energy-momentum tensor, the genuine 5-dimensional gravitational effect is contained in the projected Weyl tensor. We denote this tensor by $E_{\mu\nu}$ (see Eq. (2.4) for its definition).

A noble feature of this formalism is that the perturbations that describe the 4-dimensional gravitational degrees of freedom are already contained in the effective 4-dimensional Einstein equations and the perturbations that additionally arise from the existence of the 5-dimensional bulk are completely described by the projected Weyl tensor mentioned above. Another advantage of the formalism is that since the 5-dimensional Weyl tensor vanishes on the anti-de Sitter background, $E_{\mu\nu}$ is a physical, gauge-invariant quantity by itself.

In this paper we neglect the tensor quadratic in the matter energy-momentum and investigate the behavior of the projected Weyl tensor in detail at the linear perturbation level. We assume the AdS bulk with the de Sitter brane as the background. On this background, the 5-dimensional Weyl tensor vanishes, hence $E_{\mu\nu} = 0$. We then assume the matter energy momentum tensor on the brane, $\tau_{\mu\nu}$ to be small and consider $E_{\mu\nu}$ induced by $\tau_{\mu\nu}$.

This paper is organized as follows. In Sec. II, we derive the general solution to the projected 5-dimensional Weyl tensor perturbation, $E_{\mu\nu}$. In Sec. III, we concentrate on the cosmological radion mode and derive the explicit form of $E_{\mu\nu}$ in terms of the radion after zero-mode truncation. The physical meaning of the radion is also discussed. In Sec. IV, we summarize the result and discuss its implications.

II. WEYL TENSOR PERTURBATION

In this section, we derive the general solution to the projected 5-dimensional Weyl tensor perturbation, $E_{\mu\nu}$. Although our prime interest is on the 3-brane in the 5-dimensional spacetime, we generalize equations for the $(n-1)$-brane in the $(n+1)$-dimensional spacetime unless it is necessary to restrict to $n = 4$. The advantage is that it makes easy to see what is affected by spacetime dimensions and what is not.

First, we summerize the basic equations of the system with two $(n-1)$-branes as the fixed points of the Z_2-symmetry in a $(n+1)$-dimensional spacetime with negative cosmological constant Λ_{n+1}. The bulk metric g_{ab} obeys the $(n+1)$-dimensional Einstein equations:

$$ {}^{(n+1)}G_{ab} + \Lambda_{n+1} g_{ab} = \kappa^2 T_{ab} \,, \tag{2.1} $$

where κ^2 is the $(n+1)$-dimensional gravitational constant and T_{ab} is assumed to be zero except on the branes. We use the latin indices for $(n+1)$-dimensions in the bulk and the Greek indices for n-dimensions on the brane. On the brane, we decompose the energy momentum tensor into the tension part, $-\sigma g_{\mu\nu}$ and the matter part, $\tau_{\mu\nu}$. Although this decomposition is arbitrary in general, below we consider the background with $\tau_{\mu\nu} = 0$, which fixes the decomposition.

As derived in [1], the effective Einstein equations on the brane is written in the form,

$$ {}^{(n)}G_{\mu\nu} + \Lambda_n q_{\mu\nu} = \kappa_n^2 \tau_{\mu\nu} + \kappa^4 \pi_{\mu\nu} - E_{\mu\nu} \tag{2.2} $$

where $q_{\mu\nu}$ is the intrinsic metric on the brane,

$$ \Lambda_n := \frac{n-2}{n}\kappa^2 \left(\Lambda_{n+1} + \frac{n}{8(n-1)}\kappa^2\sigma^2 \right), \quad \kappa_n := \frac{n-2}{4(n-1)}\sigma\kappa^4, \tag{2.3} $$

$\pi_{\mu\nu}$ is a tensor quadratic in $\tau_{\mu\nu}$ whose explicit form is unnecessary, and

$$ E_{\mu\nu} := {}^{(n+1)}C^{a}{}_{\mu b\nu} n_a n^b \tag{2.4} $$

is the projected $(n+1)$-dimensional Weyl tensor where n^a is the vector unit normal to the brane.

For appropriate choice of the brane tensions, with $\tau_{\mu\nu} = \pi_{\mu\nu} = E_{\mu\nu} = 0$, the Randall-Sundrum (RS) branes [3,4] and the de Sitter branes discussed by Garriga and Sasaki [7] are solutions to these equations. The bulk spacetime of these solutions is anti-de Sitter (AdS^{n+1}), and their metric can be concisely expressed as

$$ \begin{aligned} ds^2 &= a(z)^2 \left(dz^2 + \gamma_{\mu\nu} dx^\mu dx^\nu \right); \\ a(z) &= \frac{\ell\sqrt{\mathcal{K}}}{\sinh\sqrt{\mathcal{K}}z}\,, \quad \ell = \left[\frac{-n(n-1)}{2\Lambda_{n+1}} \right]^{1/2}, \end{aligned} \tag{2.5} $$

where $\gamma_{\mu\nu}$ is the metric of a Lorentzian n-dimensional constant curvature space with curvature $\mathcal{K}=0$ or $\mathcal{K}=+1$, and ℓ is the curvature radius of the AdS^{n+1}. Then, placing two branes with positive and negative tensions at $z=z_\pm$ ($0<z_+<z_-$), and gluing two copies of the region $z_+\le z\le z_-$ together, we obtain the RS flat branes and the de-Sitter branes for $\mathcal{K}=0$ and $\mathcal{K}=+1$, respectively, in the AdS bulk. The tensions on these branes are given by

$$\sigma_\pm = \pm\frac{2(n-1)}{\kappa^2\ell}\cosh\sqrt{\mathcal{K}}z_\pm\,. \tag{2.6}$$

The n-dimensional cosmological constant Λ_n on each of the branes is given by

$$\Lambda_n = \frac{1}{2}\frac{(n-1)(n-2)\mathcal{K}}{a_\pm^2}\,, \tag{2.7}$$

where $a_\pm = a(z_\pm)$.

Now we consider the perturbation of these solutions. We assume $\tau_{\mu\nu}$ to be of $O(\epsilon)$ and consider the linear response of $E_{\mu\nu}$ to $\tau_{\mu\nu}$. We can consistently neglect the tensor $\pi_{\mu\nu}$ at the linear perturbation level.

The linear perturbation equation for $E_{\mu\nu}$ is derived in [21] on the RS flat brane background. Here we generalize it to the de Sitter brane background. For this purpose, it is useful to introduce the scaled $E_{\mu\nu}$ by

$$\hat{E}_{\mu\nu} = a^{n-2}E_{\mu\nu}. \tag{2.8}$$

Then, linearizing the equations given in [1], the equation of motion for $\hat{E}_{\mu\nu}$ in the bulk is found as

$$\left[a^{n-3}\frac{\partial}{\partial z}\frac{1}{a^{n-3}}\frac{\partial}{\partial z}+\Box_n - nK\right]\hat{E}_{\mu\nu}=0 \tag{2.9}$$

where $\Box_n := \gamma^{\mu\nu}D_\mu D_\nu$ is the n-dimensional d'Alembertian with respect to the metric $\gamma_{\mu\nu}$, and D_μ is the covariant derivative. From the Israel junction condition and the Z_2-symmetry, the boundary conditions at the branes are given by

$$\partial_z\hat{E}_{\mu\nu}\Big|_{z=z_\pm} = \pm\frac{\kappa^2}{2}a_\pm^{n-3}\left(\Sigma^T_{\mu\nu}+\Sigma^S_{\mu\nu}\right)\,, \tag{2.10}$$

where

$$\Sigma^T_{\mu\nu} = \left[-\Box_n + nK\right]\left(\tau_{\mu\nu}-\frac{a^2}{n}\gamma_{\mu\nu}\tau\right), \tag{2.11}$$

$$\Sigma^S_{\mu\nu} = -\frac{a^2}{n-1}\left[D_\mu D_\nu - \frac{1}{n}\gamma_{\mu\nu}\Box_n\right]\tau\,, \tag{2.12}$$

with $\tau := q^{\mu\nu}\tau_{\mu\nu} = a^{-2}\gamma^{\mu\nu}\tau_{\mu\nu}$.

To solve these equations for $\hat{E}_{\mu\nu}$, we introduce the Green function which satisfies

$$\begin{aligned}\left[-a^{n-3}\frac{\partial}{\partial z}\frac{1}{a^{n-3}}\frac{\partial}{\partial z}-\Box_n+n\mathcal{K}\right]&G_{\mu\nu}(x,z;x',z')^{\alpha\beta}\\ &= a^{n-3}\frac{\delta^n(x-x')}{\sqrt{-\gamma}}\delta(z-z')\left(\delta_\mu^{(\alpha}\delta_\nu^{\beta)}-\frac{1}{n}\gamma_{\mu\nu}\gamma^{\alpha\beta}\right);\\ \partial_z G_{\mu\nu}{}^{\alpha\beta}\big|_{z=z_\pm}=0.&\end{aligned} \tag{2.13}$$

The conventional causal boundary condition would be to choose the retarded Green function. For our discussions below, however, it is unnecessary to specify the causal boundary condition. Hence we leave it unspecified. The only restriction is that all Green functions appearing throughout the paper should have the same causal boundary condition. Then, the formal solution is given by

$$\begin{aligned}\hat{E}_{\mu\nu}(x,z) &= \int d^nx'\sqrt{-\gamma(x')}\left[\frac{1}{a(z')^{n-3}}G_{\mu\nu}(x,z;x'z')^{\alpha\beta}\frac{\partial}{\partial z'}\hat{E}_{\alpha\beta}(x',z')\right]_{z_+}^{z_-}\\ &= -\frac{\kappa^2}{2}\int d^nx'\sqrt{-\gamma(x')}\Big\{G_{\mu\nu}(x,z;x'z')^{\alpha\beta}\left(\Sigma^T_{\alpha\beta}(x',z')+\Sigma^S_{\alpha\beta}(x',z')\right)\Big\}_{z_+}^{z_-}\,,\end{aligned} \tag{2.14}$$

where $\left[Q(z')\right]_{z_+}^{z_-} = Q(z_-) - Q(z_+)$ and $\left\{Q(z')\right\}_{z_+}^{z_-} = Q(z_-) + Q(z_+)$.

Because the scale factor a of the bulk metric depends only on z and because the branes are located at the fixed values of z, the Green function can be expressed in the factorized form in terms of appropriately normalized mode functions. Let $\hat{E}_{\mu\nu} = \psi_m(z) Y_{\mu\nu}^{(m)}(x)$. Then Eq. (2.9) separates as

$$\left[-a^{n-3}\frac{d}{dz}\frac{1}{a^{n-3}}\frac{d}{dz} + (n-2)\mathcal{K}\right]\psi_m = m^2\psi_m\,; \qquad \frac{d}{dz}\psi_m\bigg|_{z=z_\pm} = 0. \tag{2.15}$$

$$\left[-\Box_n + m^2 + 2\mathcal{K}\right] Y_{\mu\nu}^{(m)} = 0\,; \quad Y_{\mu}^{(m)\mu} = D^\nu Y_{\mu\nu}^{(m)} = 0. \tag{2.16}$$

The n-dimensional mode function $Y_{\mu\nu}^{(m)}(x)$ may be further decomposed in terms of the $(n-1)$-dimensional spatial eigenfunctions (i.e., spatial harmonic functions). Since our interest is in the z-dependence of the mode functions, we simply denote the n-dimensional mode function by $Y_{\mu\nu}^{(m,k)}$ where k represents the $(n-1)$ sets of spatial eigenvalues. Nevertheless, it is worth noting that the spatial $SO(n)$-symmetry of the $(n-1)$-dimensional space can be used to separate the components of $Y_{\mu\nu}^{(m,k)}$ by their properties under the symmetry transformations. One then obtain the scalar, vector and tensor type mode functions with respect to the $SO(n)$-symmetry. This corresponds to the decomposition of the perturbation into the scalar, vector and tensor perturbations in the context of cosmology on the brane, i.e., in the sense of the $(n-1)$-dimensions.

From Eq. (2.15), we see that there is a mode ψ_m =constant with $m^2 = (n-2)\mathcal{K}$ that trivially satisfies the equation. Setting $\psi_m = a^{(n-3)/2} f_m$ and substituting $a(z) = \ell\sqrt{\mathcal{K}}/\sinh\sqrt{\mathcal{K}}z$, we find

$$\left[-\frac{d^2}{dz^2} + \frac{(n-5)(n-3)\mathcal{K}}{2\sinh^2\sqrt{\mathcal{K}}z} + \left(\frac{n-1}{2}\right)^2\mathcal{K}\right] f_m = m^2 f_m\,. \tag{2.17}$$

Thus when the negative-tension brane is absent (or $z_- \to \infty$), the spectrum will be continuous for $m^2 \geq [(n-1)/2]^2\mathcal{K}$. Since the mode $m^2 = (n-2)\mathcal{K}$ translates to $f_m \propto a^{-(n-3)/2}$, and it does not have a node, it is the lowest eigenmode. Since

$$\left(\frac{n-1}{2}\right)^2 - (n-2) = \left(\frac{n-3}{2}\right)^2 \geq 0, \tag{2.18}$$

the mode $m^2 = (n-2)\mathcal{K}$ is isolated from the other modes for $n > 3$. Although not necessary for the following arguments, we assume there is no mode in the range $(n-2)\mathcal{K} < m^2 < [(n-1)/2]^2\mathcal{K}$. This is the case when $n = 4$.

The orthonormality of the eigenfunction $\psi_m(z)$ of Eq. (2.15) is determined by requiring the equation to be self-adjoint. This gives

$$\int_{z_+}^{z_-} dz\, a^{3-n}\psi_m(z)\psi_{m'}(z) = \frac{N_{m^2}}{\ell^{n-3}}\,\delta_{mm'} \quad \text{for} \quad m^2 \geq \left(\frac{n-1}{2}\right)^2\mathcal{K}\,, \tag{2.19}$$

$$\int_{z_+}^{z_-} dz\, a^{3-n} = \frac{N_{m^2}}{\ell^{n-3}} \qquad \text{for} \quad m^2 = (n-2)\mathcal{K}\,, \tag{2.20}$$

where N_{m^2} is a normalization constant. In the case of $n = 4$,

$$N_{(n-2)\mathcal{K}} = \frac{\cosh\sqrt{\mathcal{K}}z_- - \cosh\sqrt{\mathcal{K}}z_+}{\mathcal{K}}\,. \tag{2.21}$$

Then we have

$$G_{\mu\nu}(x,z;x',z')^{\alpha\beta} = \ell^{n-3}\left(\frac{1}{N_{(n-2)\mathcal{K}}} G_{\mu\nu}^{((n-2)\mathcal{K})}(x,x')^{\alpha\beta} + \sum_{m\geq m_c}\frac{\psi_m(z)\psi_m(z')}{N_{m^2}} G_{\mu\nu}^{(m^2)}(x,x')^{\alpha\beta}\right)\,, \tag{2.22}$$

where $m_c^2 = \left((n-1)/2\right)^2\mathcal{K}$, and $G_{\mu\nu}^{(m^2)}(x,x')^{\alpha\beta}$ is the n-dimensional part of the Green function satisfying

$$[-\Box_n + m^2 + 2\mathcal{K}] G_{\mu\nu}^{(m^2)}(x,x')^{\alpha\beta} = \frac{\delta^n(x-x')}{\sqrt{-\gamma}}\left(\delta_\mu^{(\alpha}\delta_\nu^{\beta)} - \frac{1}{n}\gamma_{\mu\nu}\gamma^{\alpha\beta}\right)\,. \tag{2.23}$$

The n-dimensional Green function is constructed from properly normalized $Y_{\mu\nu}^{(m,k)}(x)$, but we do not give the explicit expression of $G_{\mu\nu}^{(m^2)}(x,x')^{\alpha\beta}$ in terms of $Y_{\mu\nu}^{(m,k)}(x)$, since it is unnecessary for our purpose.

Before concluding this section, we note an important implication of the above result to cosmological perturbations of the de Sitter brane-universe. Since $E_{\mu\nu}$ describes all the extra-dimensional effect on the brane-world, and since it contains all the scalar, vector and tensor type perturbations in the sense of cosmological perturbations as mentioned earlier,* we see immediately that the spectrum of the Kaluza-Klein modes is the same for all the types of cosmological perturbations, except for the lowest (zero) mode $m^2 = (n-2)\mathcal{K}$ on which we discuss in the next section. This is in agreement with the previous results on cosmological tensor [7,15] and vector [16] Kaluza-Klein spectra on the de Sitter brane.

III. COSMOLOGICAL RADION

In this section, we consider the so-called zero-mode truncation. That is, we neglect all the Kaluza-Klein modes $m^2 \geq m_c^2 = [(n-1)/2]^2\mathcal{K}$. In our formalism, this means to consider only the mode $m^2 = (n-2)\mathcal{K}$. The peculiarity of this mode is that the corresponding n-dimensional tranceverse-traceless tensor mode degenerates with the scalar mode of the eigenvalue $m_s^2 = -n\mathcal{K}$ [22]. Thus one expects this mode to play the role of the radion. In passing, it may be worthwhile to mention that the tranceverse-traceless tensor mode of the eigenvalue $m^2 = 0$ degenerates with the transverse vector mode of the eigenvalue $m_v^2 = (n-1)K$. However, since the mode $m^2 = 0$ is absent in the spectrum of $E_{\mu\nu}$ for $n > 3$, we readily see that there exists no extra radion-like vector mode in the de Sitter brane-world.

We approximate the Green function as

$$G_{\mu\nu}(x,z;x',z')^{\alpha\beta} = \frac{\ell^{n-3}}{N_{(n-2)\mathcal{K}}} G_{\mu\nu}^{((n-2)\mathcal{K})}(x,x')^{\alpha\beta} \,. \tag{3.1}$$

Then from Eq. (2.14), $\hat{E}_{\mu\nu}$ is given by

$$\hat{E}_{\mu\nu}(x) = -\frac{\kappa^2 \ell^{n-3}}{2N_{(n-2)\mathcal{K}}} \int d^n x' \sqrt{-\gamma} G_{\mu\nu}^{((n-2)\mathcal{K})}(x,x')^{\alpha\beta} \left\{ \Sigma^T_{\alpha\beta}(x') + \Sigma^S_{\alpha\beta}(x') \right\}_{z_+}^{z_-} \,. \tag{3.2}$$

Incidentally, the truncated $\hat{E}_{\mu\nu}$ turns out to be independent of the extra-dimensional coordinate z. Noting the form of the source in Eqs. (2.11) and (2.12), and the fact that the Green function obeys

$$[-\Box_n + n\mathcal{K}] G_{\mu\nu}^{((n-2)\mathcal{K})}(x,x')^{\alpha\beta} = \frac{\delta^n(x-x')}{\sqrt{-\gamma}} \left(\delta_\mu^{(\alpha}\delta_\nu^{\beta)} - \frac{1}{n}\gamma_{\mu\nu}\gamma^{\alpha\beta} \right) \,, \tag{3.3}$$

we find it is convenient to introduce the following two auxiliary n-dimensional fields. Let $\phi_{\mu\nu}(x)$ and $\phi(x)$ be the n-dimensional tensor and scalar fields which satisfy

$$[-\Box_n + n\mathcal{K}]\,\phi_{\mu\nu} = \frac{\kappa^2}{2}\Sigma^S_{\mu\nu} \,, \tag{3.4}$$

$$[-\Box_n - n\mathcal{K}]\,\phi = -\frac{\kappa^2 a^2}{2(n-1)}\,\tau \,, \tag{3.5}$$

respectively. Then Eq. (3.4) is rewritten as

$$\begin{aligned} [-\Box_n + n\mathcal{K}]\,\phi_{\mu\nu} &= -\left[D_\mu D_\nu - \frac{1}{n}\gamma_{\mu\nu}\Box_n\right]\frac{\kappa^2 a^2}{2(n-1)}\tau = \left[D_\mu D_\nu - \frac{1}{n}\gamma_{\mu\nu}\Box_n\right][-\Box_n - n\mathcal{K}]\,\phi \\ &= [-\Box_n + n\mathcal{K}]\left[D_\mu D_\nu - \frac{1}{n}\gamma_{\mu\nu}\Box_n\right]\phi \,. \end{aligned} \tag{3.6}$$

Hence, with an appropriate causal boundary condition imposed on the fields, we obtain

$$\phi_{\mu\nu} = \left[D_\mu D_\nu - \frac{1}{n}\gamma_{\mu\nu}\Box_n\right]\phi \,. \tag{3.7}$$

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*Note that the terms, scalar, vector and tensor, here are *not* used in the n-dimensional sense, but in the $(n-1)$-dimensional sense.

Using these $\phi_{\mu\nu}$ and ϕ, we obtain from Eq. (3.2)

$$
\begin{aligned}
\hat{E}_{\mu\nu}(x) &= -\frac{\kappa^2 \ell^{n-3}}{2N_{(n-2)\mathcal{K}}} \int d^n x' \sqrt{-\gamma} G_{\mu\nu}^{((n-2)\mathcal{K})}(x,x')^{\alpha\beta} \left[-\Box_n + n\mathcal{K}\right] \left\{ \tau_{\alpha\beta} - \frac{1}{n}\gamma_{\alpha\beta}\tau + \frac{2}{\kappa^2}\phi_{\mu\nu} \right\}_{z_+}^{z_-} \\
&= -\frac{\ell^{n-3}}{2N_{(n-2)\mathcal{K}}} \left\{ \kappa^2 \bar{\tau}_{\mu\nu} + 2\left[D_\mu D_\nu - \frac{1}{n}\gamma_{\mu\nu}\Box_n \right]\phi \right\}_{z_+}^{z_-}
\end{aligned}
\tag{3.8}
$$

where $\bar{\tau}_{\mu\nu} := \tau_{\mu\nu} - \frac{a^2}{n}\gamma_{\mu\nu}\tau$ and the scalar field ϕ satisfies Eq. (3.5). Thus $\hat{E}_{\mu\nu}$ in the zero-mode truncation is given by the two parts; one given in terms of $\tau_{\mu\nu}$ directly, and the other determined by the scalar field ϕ. This is our main result. For $n = 4$ and $\mathcal{K} = 0$, we recover the result obtained by Garriga and Tanaka [7]. We may therefore identify ϕ (to be precise $\left\{\phi(x)\right\}_{+}^{-}$) as the radion that describes the relative displacement of the branes.

To confirm this identification, we derive the second order action for the metric perturbation $h_{\mu\nu}$ in a gauge $h_{ab}n^b = 0$ explicitly in Appendix A. There the action is expressed in terms of the metric perturbation $h_{\mu\nu}$ in the bulk and the coordinate displacement of the brane φ. Then specializing the gauge to the Randall-Sundrum gauge in which $h_{ab}n^b = D^\mu h_{\mu\nu} = h^\mu{}_\mu = 0$, the equation of motion of φ is found as

$$
\left[-\Box_n - n\mathcal{K}\right]\varphi(x) = -\frac{\kappa^2 a^2}{2(n-1)}\tau\,. \tag{3.9}
$$

See Eq. (A19). Thus the scalar field $\phi(x)$ introduced here obeys exactly the same equation as $\varphi(x)$, and can be identified as the displacement of the brane in the Randall-Sundrum gauge.

It is instructive to consider the physical meaning of ϕ from the view point of an observer on the brane. For this purpose, we take the gaussian normal coordinates around the brane, defined by $h_{ab}n^b = 0$ and $\varphi(x) = 0$. The metric perturbation in this gauge describes what one can observe on the brane. In this gauge, we may decompose the metric perturbation as

$$
h_{\mu\nu} = a^2\left[\mathcal{R}\gamma_{\mu\nu} + D_\mu D_\nu H_T + D_{(\mu}X_{\nu)} + X_{\mu\nu}\right], \tag{3.10}
$$

where X_μ is transverse; $D^\mu X_\mu = 0$, and $X_{\mu\nu}$ is transverse-traceless; $D^\mu X_{\mu\nu} = X^\mu{}_\mu = 0$. The n-dimensional scalar potential $\mathcal{R}$ describes the scalar curvature perturbation of the brane [22]. Then the equation of motion of $\varphi(x)$, Eq. (A19), reduces to

$$
\left[-\Box_n - n\mathcal{K}\right]\frac{\mathcal{R}}{H} = -\frac{\kappa^2 a^2}{2(n-1)}\tau\,, \tag{3.11}
$$

where $H = -\partial_z a(z)/a^2(z)$. Thus $\phi(x)$ in Sec. III obeys exactly the same equation as $\mathcal{R}/H$ in the gaussian normal coordinates. That is, ϕ itself is not an additional (gravitational) scalar field on the brane, but simply corresponds to the intrinsic scalar curvature of the brane. It is important to note here that the above equation (3.11) is nothing but the trace of the linear perturbation of the effective Einstein equations (2.2) in the present background spacetime.

Thus it is not quite adequate to call ϕ the radion. The term, radion, should be used to describe the combined effect of the intrinsic scalar curvature perturbation of the brane and its effect on the non-trivial Weyl curvature perturbation $E_{\mu\nu}$ in the bulk which reacts back on the brane. In the two-brane system, this is indeed the case, and the scalar field ϕ concisely describes the effect.

For the single positive-tension brane case, we just need to discard the source term on the negative-tension brane in the result Eq. (3.8) and take the limit $z_- \to \infty$ in the above argument. Note that $z \to \infty$ corresponds to the center of the $(n-1)$-sphere in the bulk n-dimensional spatial hypersurface in the case of $\mathcal{K} = 1$. See Eqs. (2.19) and (2.20). Then, for $n = 4$ case, we find that $N_{(n-2)\mathcal{K}} \to \infty$ from Eq. (2.21), and thus $\hat{E}_{\mu\nu} \to 0$ from Eq. (3.8). That is, if there is only one positive-tension brane, the radion is absent and the gravity on the de Sitter brane reduces to the conventional 4-dimensional gravity with corrections solely from the Kaluza-Klein modes.

IV. SUMMARY AND DISCUSSION

In this paper, we have investigated the role of the radion mode on the de Sitter brane, using the covariant curvature tensor formalism developed in [1]. We have found that if there is only one positive-tension de Sitter brane, the Einstein gravity is recovered on the brane without any corrections other than those from the Kaluza-Klein modes. There is no trace of the radion mode, at least at the linear order, contrary to our naive expectation that the displacement of the brane does have physical meaning for the de Sitter 3-sphere brane and hence that some trace of it should appear on the brane. If we recall the case of a vacuum bubble in the 4-dimensional spacetime [23], we find exactly the same

phenomenon there; an observer on the wall cannot detect fluctuations of the wall, though the brane does fluctuate in the 4-dimensional sense.

To investigate the physics behind this phenomenon further, as well as to clarify the role of the scalar field ϕ, we have derived the action for the displacement of the brane, φ, coupled to the metric perturbation, $h_{\mu\nu}$, in Appendix A. The equation of motion of φ is then found to be exactly the same as that of ϕ in the Randall-Sundrum gauge. This result may be interpreted as follows. The brane does fluctuate in this gauge by the amount $\varphi(x)$ in the $(n+1)$-dimensional sense. However, the displacement of the brane cannot be detected by itself because one can always choose a local coordinate system in which $\varphi = 0$. From the view point of an observer on the brane, the natural coordinate associated with it is the gaussian normal coordinates defined by $h_{ab}n^b = 0$ and $\varphi = 0$. We have then found ϕ describes the intrinsic n-dimensional scalar curvature perturbation of the brane. Hence the effect due to the fluctuation of the brane cannot be detected unless it causes a non-trivial metric perturbation in the bulk and propagates back to the brane (or to the other brane in the two-brane case). Since the physical metric perturbation in the bulk is described by $E_{\mu\nu}$, the fact that the Green function in the zero-mode truncation, Eq. (3.1), vanishes in the limit of a single brane implies that the bulk metric becomes increasingly rigid as the location of the negative-tension brane approaches $z_- \to \infty$ and the fluctuation ceases to propagate through the bulk. As a result, the gravity on the positive-tension brane is not affected by the fluctuation of the brane, $E_{\mu\nu} = 0$, in the single brane case.

In the two-brane system, on the other hand, there exists the cosmological radion mode as in the flat two-brane system. An interesting fact is that the cosmological radion has the negative mass-squared $m^2 = -n\mathcal{K}$. This may have important implications to the cosmological perturbation theory. In connection with this issue, we should mention that our result that the radion has the negative mass-squared is in apparent contradiction with the result obtained by Chiba [24], who derived a non-linear effective action for the radion mode by assuming a certain form of the 5-dimensional metric. Naively applying his result to curved background and taking the large separation limit, one finds the radion is a massless conformal scalar, i.e., it effectively has a positive mass-squared on the de Sitter brane. However, it should be noted that the assumed form of the 5-dimensional action used in [24] is valid only for the flat brane background. So, there is in fact no logical contradiction. To understand the role of the radion in the inflationary brane universe, it will be necessary to derive the effective action of the radion on the de Sitter brane. This issue is currently under investigation.

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APPENDIX A: THE SURFACE TERM ACTION FOR A THIN-WALL SYSTEM

In this appendix, we consider an n-dimensional thin-wall (i.e., $(n-1)$-brane) system in the $(n+1)$-dimensional bulk spacetime and derive the perturbative second order action for the displacement of the brane. The system consists of two manifolds $\mathcal{M}_+$ and $\mathcal{M}_-$ with a common boundary $\partial\mathcal{M}_+ = \partial\mathcal{M}_- = \Sigma$ on which the tension σ and matter fields with the Lagrangian $\mathcal{L}_m$ are present. The background metric is assumed to be of the form, $ds^2 = dr^2 + q_{\mu\nu}dx^\mu dx^\nu$ with the surface Σ at $r = r_0$. For most of the discussions below, we shall not impose the Z_2-symmetry across Σ. Hence, the result can be applied to a general thin-wall system.

The action of the system is given by

$$
\begin{aligned}
I = I_{R\text{-}2\Lambda,+} + I_{R\text{-}2\Lambda,-} + I_{K,+} - I_{K,-} + I_\sigma + I_m\,;& \\
I_{R\text{-}2\Lambda,\pm} := \frac{1}{2\kappa^2}\int_{\mathcal{M}_\pm}\sqrt{-g}drd^nx\,(R-2\Lambda)\,, \qquad I_{K,\pm} &:= \frac{1}{\kappa^2}\int_{\partial\mathcal{M}_+}\sqrt{-q}d^nx\,K\,, \\
I_\sigma := -\int_\Sigma\sqrt{-q}d^nx\,\sigma\,, \qquad I_m := \frac{1}{2}\int_\Sigma\sqrt{-q}d^nx\,\mathcal{L}_m\,,&
\end{aligned}
$$

where $q_{ab} = g_{ab} - n_a n_b$, $K_{ab} := q_a{}^c q_b{}^d \nabla_c n_d$ and n^a is defined such that it points from $\mathcal{M}_+$ to $\mathcal{M}_-$. The variation of this action with respect to the bulk gravitational field gives the Einstein equations in $\mathcal{M}_+$ and $\mathcal{M}_-$ and that with respect to the metric on Σ gives the Israel junction condition. Our strategy to obtain the second order action is as follows. Apart from simple terms which can be expanded to second order easily, we consider the surface term δI_Σ arising from the variation of the action by assuming the bulk gravitational equations are satisfied. Then considering the displacement $r_0 \to r_0 \pm \varphi_\pm(x)$ of the boundary surface and the metric perturbation $q_{\mu\nu} \to q_{\mu\nu} + h_{\mu\nu}$, we integrate the variation δI_Σ to obtain the second order action on Σ.

The action may be rewritten as

$$I = \left\{ \frac{1}{2\kappa^2} \int_{\mathcal{M}} dr d^n x \, \mathcal{I}_{R\Lambda} \right\}_-^+ + \frac{1}{\kappa^2} \int_{\Sigma} d^n x \, \mathcal{I}_{K\sigma} + I_m \, ;$$

$$\mathcal{I}_{R\Lambda} := \sqrt{-g}\Big[R - 2\Lambda \Big] , \quad \mathcal{I}_{K\sigma} := \sqrt{-q}\Big[\Delta K - \sigma\kappa^2 \Big] , \tag{A1}$$

where $\left\{ \int_{\mathcal{M}} \right\}_-^+ = \int_{\mathcal{M}_+} + \int_{\mathcal{M}_-}$, $\Delta Q := Q_+ - Q_-$ and $Q_\pm$ is a quantity on Σ calculated from the $\mathcal{M}_\pm$ sides, respectively. The variation of the action is given by

$$\delta I = \left\{ \frac{1}{2\kappa^2} \int_{\mathcal{M}} dr \, d^n x \, \frac{\delta \mathcal{I}_{R\Lambda}}{\delta g_{ab}} \, \delta g_{ab} \right\}_-^+ + \left\{ \frac{1}{2\kappa^2} \int_{\Sigma} d^n x \, \mathcal{I}_{R\Lambda} \delta r \right\}_-^+$$
$$+ \frac{1}{\kappa^2} \int_{\Sigma} d^n x \left[\Delta \pi^{ab} + \frac{\delta \mathcal{I}_{K\sigma}}{\delta q_{ab}} \right] \delta q_{ab} + \frac{1}{\kappa^2} \delta_\Sigma \left(\int_{\Sigma} d^n x \, \mathcal{I}_{K\sigma} \right)$$
$$+ \frac{1}{2} \int_{\Sigma} \sqrt{-q} d^n x \, \tau^{\mu\nu} \delta q_{\mu\nu} + \frac{1}{2} \delta_\Sigma \left(\int_{\Sigma} \sqrt{-q} d^n x \, \mathcal{L}_m \right) , \tag{A2}$$

where δr describes the variation of the domain of integration $\delta\mathcal{M}$, π^{ab} is the boundary term that appears from $\mathcal{I}_{R\Lambda}$ by the variation of the metric, and δ_Σ denotes the variation with respect to the displacement of the boundary surface without changing the metric on the manifolds $\mathcal{M}_\pm$. Note here

$$\Delta \pi^{ab} + \frac{\delta \mathcal{I}_{K\sigma}}{\delta q_{ab}} = \Delta K^{ab} - \Delta K q^{ab} + \sigma \kappa^2 q^{ab} . \tag{A3}$$

Now we consider the perturbation. We choose the 'synchronous' coordinates with respect to the extra-dimensional direction. So, the metric in $\mathcal{M}_\pm$ takes the form,

$$ds_\pm^2 = dr_\pm^2 + (\bar{q}_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu \, , \tag{A4}$$

in which the unperturbed boundary $\bar{\Sigma}$ is at $r_\pm = r_0$ and both $r_\pm$ increase towards $\bar{\Sigma}$, $\bar{q}_{\mu\nu}$ is the background metric, and $h_{\mu\nu}$ is the perturbation of the metric. Here and in what follows, barred quantities denote background quantities. The perturbed boundary Σ is placed at $r_\pm = r_0 \pm \varphi_\pm(x)$. The perturbation of the surface term is most conveniently calculated by introducing the new coordinates defined by $\hat{r} := r \mp \varphi_\pm(x)$ and $\hat{x}^\mu := x^\mu$ in which the $\hat{r}$-coordinate of the surface Σ is unperturbed at $\hat{r} = r_0$. Then the metric takes the form,

$$ds^2 = d\hat{r}^2 + 2\varphi_{|\mu} d\hat{r} d\hat{x}^\mu + \left[\bar{q}_{\mu\nu} + h_{\mu\nu} + \varphi_{|\mu}\varphi_{|\nu} \right] d\hat{x}^\mu d\hat{x}^\nu \, ,$$
$$g^{ab} = (1 + \varphi_{|\rho}\varphi^{|\rho})(\partial_{\hat{r}})^a (\partial_{\hat{r}})^b - 2[\varphi^{|\mu} - h^{\mu\nu}\varphi_{|\nu}](\partial_{\hat{r}})^{(a}(\partial_\mu)^{b)} + \bar{q}^{\mu\nu}(\partial_\mu)^a(\partial_\nu)^b \, . \tag{A5}$$

Hence, the vector n^a unit normal to the boundary Σ at $\hat{r} = r_0$ is given by

$$n^a = (1 + \frac{1}{2}\varphi_{|\rho}\varphi_{|\rho})(\partial_{\hat{r}})^a - [\varphi^{|\mu} - h^{\mu\nu}\varphi_{|\nu}](\partial_\mu)^a . \tag{A6}$$

The metric on the boundary and its determinant are given by

$$ds_\Sigma^2 = [\bar{q}_{\mu\nu} + h_{\mu\nu} + \varphi_{|\mu}\varphi_{|\nu}] d\hat{x}^\mu d\hat{x}^\nu \, ,$$
$$\sqrt{-q(\hat{x})} = \sqrt{-\bar{q}} \left[1 + \frac{1}{2}(h + \varphi_{|\rho}\varphi^{|\rho}) + \frac{1}{8}(h^2 - 2h^{\rho\sigma}h_{\rho\sigma}) \right] . \tag{A7}$$

The extrinsic curvature of the boundary surface is given by

$$K_{\mu\nu} = \bar{K}_{\mu\nu} + k_{\mu\nu} - \varphi_{|(\mu\nu)} + \frac{1}{2}\varphi_{|\rho}\varphi^{|\rho}\bar{K}_{\mu\nu} \, ; \quad k_{\mu\nu} := \frac{1}{2}\partial_r h_{\mu\nu} \, , \tag{A8}$$

Now, we calculate the perturbation of the surface term in Eq. (A2). The first term in Eq. (A2) does not contribute to the surface term. The second term gives

$$I_{\delta\mathcal{M}} := \left\{ \frac{1}{2\kappa^2} \int_{\delta r} dr \int_{\Sigma} d^n x \, \mathcal{I}_{R\Lambda} \right\}_-^+$$
$$= \frac{1}{2\kappa^2} \int_{r_0}^{r_0 + \varphi_+} dr \, d^n x \, \sqrt{-\bar{g}} \left[\bar{R} - 2\Lambda \right] + \frac{1}{2\kappa^2} \int_{r_0}^{r_0 - \varphi_-} dr \, d^n x \, \sqrt{-\bar{g}} \left[\bar{R} - 2\Lambda \right]$$
$$= \frac{1}{\kappa^2} \int_{\bar{\Sigma}} d^n x \sqrt{-\bar{q}} \left[\frac{2\Lambda}{n-1} \left(\Delta\varphi + \frac{1}{2}\mathcal{K}[\varphi_+^2 + \varphi_-^2] \right) \right] , \tag{A9}$$

in which the trace of the background Einstein equations $\bar{R} = 2\Lambda(n+1)/(n-1)$ is used. The third term is the term that gives the Israel junction condition in the absence of matter fields. See Eq. (A3). Hence, by considering the perturbed metric (A7) and extrinsic curvature (A8), and identifying $\delta q_{\mu\nu} = \delta h_{\mu\nu}$, we find

$$\begin{aligned}
\delta_h I_\pi^{(2)} &:= \int_\Sigma d^n x \Big[\Delta\pi^{ab} + \frac{\delta\mathcal{I}_{K\sigma}}{\delta q_{ab}}\Big]\delta h_{ab} \\
&= \delta_h\Big\{ -\frac{1}{2}\int_\Sigma d^n x\sqrt{-\bar{q}}\Big(\Big[\Delta\bar{K}^{ab} - \Delta\bar{K}\bar{q}^{ab} + \sigma\kappa^2\bar{q}^{ab}\Big]h_{\mu\nu} + {}^{(2)}\mathcal{I}_J[h,\varphi]\Big)\Big\} \\
&= \delta_h\Big\{ -\frac{1}{2}\int_{\bar{\Sigma}} d^n x\sqrt{-\bar{q}}\Big(h_{\mu\nu}\Delta(C_1^{\mu\nu}\varphi) + {}^{(2)}\mathcal{I}_J[h,\varphi]\Big)\Big\},
\end{aligned}$$

where

$$\begin{aligned}
{}^{(2)}\mathcal{I}_J[h,\varphi] &:= \Delta k^{\mu\nu}h_{\mu\nu} - \Delta k_\sigma{}^\sigma h + \frac{\sigma\kappa^2}{2(n-1)}(h^2 - h^{\mu\nu}h_{\mu\nu}) - \Delta\varphi^{|(\mu\nu)}h_{\mu\nu} + h\Box\Delta\varphi, \\
C_1^{\mu\nu} &:= \partial_r\bar{K}^{\mu\nu} - \bar{q}^{\mu\nu}\partial_r\bar{K} + \frac{2}{n}\bar{K}\bar{K}^{\mu\nu}, \qquad \text{(A10)}
\end{aligned}$$

and $\Box = \bar{q}^{\mu\nu}D_\mu D_\nu$. The C_1 term comes from the displacement of the surface Σ from $\bar{\Sigma}$ in the original coordinates (r, x^μ). Thus we obtain

$$I_\pi^{(2)} = -\frac{1}{2\kappa^2}\int_{\bar{\Sigma}} d^n x\sqrt{-\bar{q}}\Big(h_{\mu\nu}\Delta(C_1^{\mu\nu}\varphi) + {}^{(2)}\mathcal{I}_J[h,\varphi]\Big)\Big\}. \qquad \text{(A11)}$$

The fourth term is the term arising from the displacement of the boundary surface Σ without altering the metric on the Manifold. This results in the changes $\bar{q}_{\mu\nu} \to \bar{q}_{\mu\nu} + \varphi_{|\mu}\varphi_{|\nu}$ and $\bar{n}^a \to \bar{n}^a - \varphi^{|\mu}(\partial_\mu)^a$, in addition to the change in the r-coordinate of Σ. Hence, the fourth term gives

$$\begin{aligned}
I_{\delta\Sigma} &:= \frac{1}{\kappa^2}\Big(\int_\Sigma d^n x\mathcal{I}_{K\sigma} - \int_{\bar{\Sigma}} d^n x\bar{\mathcal{I}}_{K\sigma}\Big) \\
&= \frac{1}{\kappa^2}\int_\Sigma d^n x\sqrt{-\bar{q}}\Big[-\frac{1}{2}\sigma\kappa^2(\varphi_+\Box\varphi_+ + \varphi_-\Box\varphi_-)\Big] \\
&\qquad\qquad + \frac{1}{\kappa^2}\Big(\int_\Sigma d^n x\sqrt{-\bar{q}}\Big[\Delta\bar{K} - \sigma\kappa^2\Big] - \int_{\bar{\Sigma}} d^n x\sqrt{-\bar{q}}\Big[\Delta\bar{K} - \sigma\kappa^2\Big]\Big) \\
&= \frac{1}{\kappa^2}\int_{\bar{\Sigma}} d^n x\sqrt{-\bar{q}}\Big[-\frac{1}{2}\sigma\kappa^2(\varphi_+\Box\varphi_+ + \varphi_-\Box\varphi_-) + \Delta(C_2\varphi) + \frac{1}{2}(C_{3,+}\varphi_+^2 + C_{3,+}\varphi_-^2)\Big]; \qquad \text{(A12)} \\
C_2 &:= \frac{1}{n}\bar{K}^2 + \partial_r\bar{K}, \quad C_3 := \frac{1}{n}\Big[(2n+1)\bar{K}\partial_r\bar{K} + \bar{K}^3 + n\partial_r^2\bar{K}\Big], \qquad \text{(A13)}
\end{aligned}$$

where Eq. (A8) with $h_{\mu\nu} = 0$ has been also used. As the C_1 term, C_2 and C_3 terms come from the displacement of the surface Σ in the original coordinates (r, x^μ). The fifth and the sixth terms give

$$\begin{aligned}
I_m^{(2)} &:= \frac{1}{2}\int_{\bar{\Sigma}}\sqrt{-q}d^n x\tau^{\mu\nu}h_{\mu\nu} + \frac{1}{2}\Big(\int_\Sigma\sqrt{-q}d^n x\mathcal{L}_m - \int_{\bar{\Sigma}}\sqrt{-\bar{q}}d^n x\mathcal{L}_m\Big) \\
&= \frac{1}{2}\int_{\bar{\Sigma}}\sqrt{-\bar{q}}d^n x\tau^{\mu\nu}h_{\mu\nu} + \frac{1}{2}\int_{\bar{\Sigma}}\frac{1}{2}\Delta\Big(\frac{\delta(\sqrt{-q}d^n x\mathcal{L}_m)}{\delta\bar{q}_{\mu\nu}}\frac{\partial\bar{q}_{\mu\nu}}{\partial r}\varphi\Big) \\
&= \frac{1}{2}\int_{\bar{\Sigma}}\sqrt{-\bar{q}}d^n x\Big[\tau^{\mu\nu}h_{\mu\nu} + \tau^{\mu\nu}\Delta(\bar{K}_{\mu\nu}\varphi)\Big]. \qquad \text{(A14)}
\end{aligned}$$

Adding up Eqs. (A9), (A11), (A13) and (A14) all together, we find the action for the surface term, $I_\Sigma[h,\varphi]$, up to second order as

$$\begin{aligned}
I_\Sigma[h,\varphi] &= I_{\delta\mathcal{M}} + I_{\delta\Sigma} + I_\pi^{(2)} + I_m^{(2)} \\
&= I_\Sigma^{(1)}[h,\varphi] + I_\Sigma^{(2)}[h,\varphi]; \\
I_\Sigma^{(1)}[h,\varphi] &:= \frac{1}{\kappa^2}\int_{\bar{\Sigma}} d^n x\sqrt{-\bar{q}}\Big[\Delta(C_2\varphi) + \frac{2\Lambda}{n-1}\Delta\varphi\Big], \qquad \text{(A15)}
\end{aligned}$$

$$I^{(2)}_{\Sigma}[h,\varphi] := \frac{1}{2\kappa^2}\int_{\bar\Sigma} d^n x\sqrt{-\bar q}\Big[-\Delta k^{\mu\nu}h_{\mu\nu}+\Delta k_\sigma{}^\sigma h-\frac{\sigma\kappa^2}{2(n-1)}(h^2-h^{\mu\nu}h_{\mu\nu})$$
$$+h_{\mu\nu}\Delta\varphi^{|\mu\nu}-h\Box\Delta\varphi-h_{\mu\nu}\Delta(C_1^{\mu\nu}\varphi)+\kappa^2\tau^{\mu\nu}\Delta(\bar K_{\mu\nu}\varphi)+\kappa^2\tau^{\mu\nu}h_{\mu\nu}$$
$$-\frac{1}{2}\sigma\kappa^2(\varphi_+\Box\varphi_++\varphi_-\Box\varphi_-)+(C_{3,+}\varphi_+^2+C_{3,-}\varphi_-^2)+\frac{2\Lambda}{n-1}\mathcal{K}(\varphi_+^2+\varphi_-^2)\Big]. \quad \text{(A16)}$$

For the brane-world system specified by Eq. (2.5) with Z_2-symmetry, we have

$$\bar K^{\mu\nu} := \bar K^{\mu\nu}_+ = -\bar K^{\mu\nu}_- = \frac{1}{2}\Delta\bar K_{\mu\nu}\,,\quad \varphi := \varphi_+ = -\varphi_- = \frac{1}{2}\Delta\varphi$$
$$\bar K^{\mu\nu} = Hq^{\mu\nu}\,,\quad \partial_r H = -a^{-2}\mathcal{K}\,,\quad \partial_r^2 H = 2Ha^{-2}\mathcal{K}\,,\quad 2\Lambda = -\ell^{-2}n(n-1)\,,$$

where $H=\partial_r a/a$, and hence,

$$C_1^{\mu\nu} = 2(n-1)a^{-2}\mathcal{K}q^{\mu\nu}\,,\quad C_2 = -4\Lambda/(n-1)\,,$$
$$C_{3,\pm} = 2n^2 H\ell^{-2} - n\sigma\kappa^2 a^{-2}\mathcal{K}\,. \quad \text{(A17)}$$

As a result, we have $I^{(1)}_\Sigma = 0$, and $I^{(2)}_\Sigma$ reduces to

$$I^{(2)}_{\Sigma}[h,\varphi] = \frac{1}{\kappa^2}\int_{\Sigma} d^n x\sqrt{-q}\Big[-k^{\mu\nu}h_{\mu\nu}+k_\sigma{}^\sigma h-\frac{1}{2}H(h^2-h^{\mu\nu}h_{\mu\nu})$$
$$+h_{\mu\nu}\varphi^{|\mu\nu}-ha^{-2}\Box_n\varphi-(n-1)a^{-2}\mathcal{K}h\varphi+\frac{1}{2}\kappa^2\tau^{\mu\nu}h_{\mu\nu}$$
$$+\frac{1}{2}a^{-2}\sigma\kappa^2\left(-\varphi\Box_n\varphi-n\mathcal{K}\varphi^2+\frac{\kappa^2a^2}{n-1}\tau\varphi\right)\Big], \quad \text{(A18)}$$

where $\Box_n = a^2\Box = \gamma^{\mu\nu}D_\mu D_\nu$.

Taking the variation of the action $I^{(2)}_\Sigma[h,\varphi]$ with respect to φ, we obtain

$$[-\Box_n - n\mathcal{K}]\,\varphi = -\frac{\kappa^2a^2}{2(n-1)}\tau+\mathcal{Q}[h]\,, \quad \text{(A19)}$$

where

$$\mathcal{Q}[h] := \frac{1}{\sigma\kappa^2}\left[-a^2 h_{\mu\nu}{}^{|\mu\nu}+(\Box_n+(n-1)\mathcal{K})h\right]. \quad \text{(A20)}$$

This equation of motion shows how the displacement φ is coupled with the metric and matter fields on the brane. Taking the variation of the action $I^{(2)}_\Sigma[h,\varphi]$ with respect to $h_{\mu\nu}$, we obtain

$$-k^{\mu\nu}+Hh^{\mu\nu}+k_\sigma{}^\sigma q^{\mu\nu}-Hhq^{\mu\nu}+\varphi^{|\mu\nu}+a^{-2}\mathcal{K}\varphi q^{\mu\nu}-a^{-2}q^{\mu\nu}(a^2\Box_n+n\mathcal{K})\varphi+\frac{1}{2}\kappa^2\tau^{\mu\nu}=0\,. \quad \text{(A21)}$$

By subtracting off the trace of the above equation and using Eq. (A19), we find

$$\partial_r h_{\mu\nu}-2Hh_{\mu\nu} = 2\left[\varphi_{|\mu\nu}+\mathcal{K}\varphi\gamma_{\mu\nu}\right]+\kappa^2\left(\tau_{\mu\nu}-\frac{1}{n-1}\tau q_{\mu\nu}\right). \quad \text{(A22)}$$

This is the perturbed junction condition in the 'synchronous' gauge; $h_{ab}n^b=0$.

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